

# FINITE SEMIGROUPS THAT ARE MINIMAL FOR NOT BEING MALCEV NILPOTENT

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ABSTRACT. We give a description of the finite semigroups that are minimal for not being Malcev nilpotent.

## 1. INTRODUCTION

For elements  $x, y, z_1, z_2, \dots$  in a semigroup  $S$  one recursively defines two sequences

$$\lambda_n = \lambda_n(x, y, z_1, \dots, z_n) \quad \text{and} \quad \rho_n = \rho_n(x, y, z_1, \dots, z_n)$$

by

$$\lambda_0 = x, \quad \rho_0 = y$$

and

$$\lambda_{n+1} = \lambda_n z_{n+1} \rho_n, \quad \rho_{n+1} = \rho_n z_{n+1} \lambda_n.$$

Recall that a semigroup is said to be *nilpotent* (in the sense of Mal'cev [9], denoted (MN) in [5]) if there exists a positive integer  $n$  such that

$$\lambda_n(a, b, c_1, \dots, c_n) = \rho_n(a, b, c_1, \dots, c_n)$$

for all  $a, b$  in  $S$  and  $c_1, \dots, c_n$  in  $S^1$ . The smallest such  $n$  is called the nilpotency class of  $S$ . A semigroup  $S$  is said to be *positively Engel* (PE) if for some positive integer  $n \geq 2$ ,

$$\lambda_n(a, b, 1, 1, c, c^2, \dots, c^{n-2}) = \rho_n(a, b, 1, 1, c, c^2, \dots, c^{n-2})$$

for all  $a, b$  in  $S$  and  $c \in S^1$ .

It is well known that a group  $G$  is nilpotent of class  $n$  if and only if it is nilpotent of class  $n$  in the classical sense. Nilpotent semigroups and their semigroup algebras have been investigated in [3, 5, 11]. For example, it is proved that a completely 0-simple semigroup  $S$  is nilpotent if and only if  $S$  is an inverse semigroup with nilpotent maximal subgroups. If  $S$  is a semigroup with a zero  $\theta$ , then obviously an ideal  $I$  with  $I^n = \{\theta\}$  is a nilpotent semigroup as well.

In [6, 7] we started with the investigations on what structure of a finite semigroup is determined by its 2-generated nilpotent subsemigroups. For

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this, in [6], we introduced the non-nilpotent graph  $\mathcal{N}_S$  associated to a finite semigroup  $S$  (earlier this was done for groups by Abdollahi and Zarrin in [1]). Recall that the vertices of  $\mathcal{N}_S$  are the elements of  $S$  and that there is an edge between  $x$  and  $y$  if the semigroup generated by  $x$  and  $y$ , denote  $\langle x, y \rangle$ , is not nilpotent. Note that  $\mathcal{N}_S$  is empty if  $S$  is a nilpotent semigroup. We mention some of the results. First, if a finite semigroup  $S$  has empty upper non-nilpotent graph then  $S$  is positively Engel, but in general it is not nilpotent. On the other hand, a semigroup has a complete upper non-nilpotent graph if and only if it is a completely simple semigroup that is a band. One of the main results states that if all connected  $\mathcal{N}_S$ -components of a semigroup  $S$  are complete (with at least two elements) then  $S$  is a band that is a semilattice of its connected components and, moreover,  $S$  is an iterated total ideal extension of its connected components. Some of the examples show that there exist finite semigroups that are not nilpotent and not generated by two elements but every proper subsemigroup is nilpotent.

In [7] we described a class of finite semigroups that are near to being nilpotent, called pseudo nilpotent semigroups.

In this paper we continue the investigations on finite semigroups that are close to being nilpotent. Obviously every finite semigroup that is not nilpotent has a subsemigroup that is minimal for not being nilpotent, i.e. every proper subsemigroup and every Rees factor semigroup is nilpotent. We simply call such a semigroup a minimal non-nilpotent semigroup.

In the case of finite groups a characterization has been given by Schmidt in [13] (see also [12, Theorem 6.5.7] or [8, Theorem Schmidt-Rtdei-Iwasawa]). Of course, a finite group that is not nilpotent is minimal non-nilpotent (we simply call them Schmidt groups) if and only if every proper subgroup is nilpotent. Such a group  $G$  has the following properties:

- (1)  $|G| = p^a q^b$  where  $p, q$  are distinct primes,  $a, b > 0$ ,  $G$  has a normal Sylow  $p$ -subgroup and the Sylow  $q$ -subgroups are cyclic.
- (2) The Frattini subgroups of Sylow subgroups of  $G$  are central in  $G$ .
- (3)  $G$  is 2-generated, i.e.  $G = \langle g_1, g_2 \rangle$  for some  $g_1, g_2 \in G$ .

For standard notations and terminology we refer to [2]. A completely 0-simple finite semigroup  $S$  is isomorphic with a regular Rees matrix semigroup  $\mathcal{M}^0(G, n, m; P)$ , where  $G$  is a maximal subgroup of  $S$ ,  $P$  is the  $m \times n$  sandwich matrix with entries in  $G^\theta$  and  $n$  and  $m$  are positive integers. The elements of  $S$  we denote by  $(g; i, j)$ , where  $g \in G^\theta$ ,  $1 \leq i \leq n$  and  $1 \leq j \leq m$ ; the zero element is simply denoted  $\theta$ . The element of  $P$  on the  $(i, j)$  position we denote by  $p_{ij}$ . The non-zero elements we denote by  $\mathcal{M}(G, n, m; P)$ . If all elements of  $P$  are non-zero then this is a semigroup and every completely simple finite semigroup is of this form. If  $P = I_n$ , the identity matrix, then  $S$  is an inverse semigroup. By what is mentioned earlier, a completely 0-simple semigroup  $\mathcal{M}^0(G, n, m; P)$  is nilpotent if and only if  $n = m$ ,  $P = I_n$  and  $G$  is a nilpotent group [3].

## 2. PROPERTIES OF MINIMAL NON-NILPOTENT SEMIGROUPS

The starting point of our investigations is the following necessary and sufficient condition for a finite semigroup not to be nilpotent [6].

**Lemma 2.1.** *A finite semigroup  $S$  is not nilpotent if and only if there exists a positive integer  $m$ , distinct elements  $x, y \in S$  and elements  $w_1, w_2, \dots, w_m \in S^1$  such that  $x = \lambda_m(x, y, w_1, w_2, \dots, w_m)$ ,  $y = \rho_m(x, y, w_1, w_2, \dots, w_m)$ .*

Recall that if  $S$  is a semigroup with ideal  $I$  such that both  $I$  and  $S/I$  are nilpotent semigroups then it does not follow in general that  $S$  is nilpotent. For counter examples we refer the reader to [3]. However, if  $I^n = \{\theta\}$  (with  $\theta$  the zero element of  $S$ ) and  $S/I$  is nilpotent then  $S$  is a nilpotent semigroup. This easily follows from the previous lemma. Another easy consequence is the following.

**Lemma 2.2.** *Let  $S$  be a semigroup. If a monoid  $T$  is a subsemigroup of  $S$  such that  $T^\theta$  is an ideal in  $S^\theta$  and there exists a positive integer  $m$ , distinct elements  $x, y \in T$  and  $w_1, w_2, \dots, w_m \in S^1$ , such that  $x = \lambda_m(x, y, w_1, w_2, \dots, w_m)$ ,  $y = \rho_m(x, y, w_1, w_2, \dots, w_m)$ , then  $T$  is not nilpotent.*

*Proof.* The identity of  $T$  is central in  $S$ . By assumption,  $T$  is an ideal of  $S$ , the result then follows at once from Lemma 2.1.  $\square$

It easily is verified that a finite semigroup of minimal cardinality that is a minimal non-nilpotent semigroup but is not a group is the band  $U_1 = \{e, f\}$  with  $ef = f$  and  $fe = e$ . It also can be described as the completely simple semigroup  $\mathcal{M}(\{e\}, \{1\}, \{1, 2\}; \begin{pmatrix} 1 \\ 1 \end{pmatrix})$ . It turns out that these are precisely the minimal non-nilpotent semigroups that are simple.

**Lemma 2.3.** *Let  $S$  be a finite semigroup. If  $S$  is minimal non-nilpotent then one of the following properties hold*

- (1)  $S$  is a minimal non-nilpotent group;
- (2)  $S = U_1$ ;
- (3)  $S$  has an ideal that is isomorphic with  $\mathcal{M}^0(G, n, n, I_n)$ , a completely 0-simple ideal over a nilpotent group that is an inverse semigroup with  $n \geq 2$ .

*Proof.* Since  $S$  is finite, it has a principal series

$$S = S_1 \supset S_2 \supset \dots \supset S_{h'} \supset S_{h'+1} = \emptyset.$$

That is, each  $S_i$  is an ideal of  $S$  and there is no ideal of  $S$  strictly between  $S_i$  and  $S_{i+1}$  (for convenience we call the empty set an ideal of  $S$ ). Each principal factor  $S_i/S_{i+1}$  ( $1 \leq i \leq m$ ) of  $S$  either is completely 0-simple, completely simple or null.

Assume  $S$  is minimal non-nilpotent. So, by Lemma 2.1, there exist distinct elements  $s_1, s_2 \in S$  and elements  $w_1, w_2, \dots, w_h \in S^1$  such that

- (1)  $s_1 = \lambda_h(s_1, s_2, w_1, w_2, \dots, w_h)$  and  $s_2 = \rho_h(s_1, s_2, w_1, w_2, \dots, w_h)$ .

Suppose that  $s_1 \in S_i \setminus S_{i+1}$ . Because  $S_i$  and  $S_{i+1}$  are ideals of  $S$ , the equalities imply that  $s_2 \in S_i \setminus S_{i+1}$  and  $w_1, w_2, \dots, w_h \in S \setminus S_{i+1}$ . Furthermore, one obtains that  $S_i/S_{i+1}$  is a completely 0-simple  $\mathcal{M}^0(G, n, m, P)$  or a completely simple  $\mathcal{M}(G, n, m, P)$ . Also, since  $S$  is minimal non-nilpotent,  $S_{i+1} = \emptyset$  or  $S_{i+1} = \{\theta\}$ .

If  $n+m = 2$  then  $S_i \setminus S_{i+1}$  is a group and  $S_i^\theta$  is an ideal of  $S^\theta$ . By Lemma 2.2,  $S_i \setminus S_{i+1}$  is not nilpotent. Hence, since  $S$  is minimal non-nilpotent,  $S = S_i \setminus S_{i+1}$  and so  $S$  is a minimal non-nilpotent group.

Suppose  $n+m > 2$ . If  $S_i/S_{i+1}$  is not nilpotent then (by the results mentioned in the introduction) we may assume that the first row of  $P$  contains two non-zero elements, say  $p_{1i}$  and  $p_{1j}$ . It is then easily verified that the subsemigroup  $\langle (p_{1i}^{-1}; i, 1), (p_{1j}^{-1}; j, 1) \rangle$  is isomorphic with the minimal non-nilpotent semigroup  $U_1$ . Hence,  $S = U_1$ .

The remaining case is  $n+m > 2$  and  $S_i/S_{i+1}$  is a nilpotent semigroup. As mentioned in the introduction, in this case  $S_i/S_{i+1} = \mathcal{M}^0(G, n, n; I_n)$  with  $G$  a nilpotent group.  $\square$

In the remaining of the paper we thus assume that  $S$  is a finite minimal non-nilpotent semigroup that has an ideal  $M = \mathcal{M}^0(G, n, n; I_n)$  with  $G$  a nilpotent group and  $n > 1$ . We obtain a representation  $\Gamma$  of  $S$  (i.e. a semigroup homomorphism) to the full transformation semigroup  $\mathcal{T}_{\{1, \dots, n\} \cup \{\theta\}}$  of the set  $\{1, \dots, n\} \cup \{\theta\}$

$$\Gamma : S \longrightarrow \mathcal{T}_{\{1, \dots, n\} \cup \{\theta\}}$$

defined as follows, for  $1 \leq i \leq n$ ,

$$\Gamma(s)(i) = \begin{cases} i' & \text{if } s(g; i, j) = (g'; i', j) \text{ for some } g, g' \in G, 1 \leq j \leq n \\ \theta & \text{otherwise} \end{cases}$$

and

$$\Gamma(s)(\theta) = \theta.$$

We call  $\Gamma$  a minimal non-nilpotent representation of  $S$  and  $\Gamma(S)$  a minimal non-nilpotent image of  $S$ .

Also, for every  $s \in S$ , we define a map

$$(2) \quad \Psi(s) : \{1, \dots, n\} \cup \{\theta\} \longrightarrow G^\theta$$

as follows

$$\Psi(s)(i) = g \quad \text{if } \Gamma(s)(i) \neq \theta \text{ and } s(1_G; i, j) = (g; \Gamma(s)(i), j)$$

for some  $1 \leq j \leq n$ , otherwise  $\Psi(s)(i) = \theta$ . It is straightforward to verify that  $\Psi$  is well-defined.

Note that if  $\Psi(s)(i) = g$  and  $g \in G$  then  $s(h; i, j) = (gh; \Gamma(s)(i), j)$  for every  $h \in G$ . Also if  $\Psi(st)(i) = g$ ,  $\Psi(t)(i) = g'$  and  $\Psi(s)(\Gamma(t)(i)) = g''$ , then  $g = g''g'$ . Hence it follows that  $\Psi(st) = (\Psi(s) \circ \Gamma(t)) \Psi(t)$ .

We claim that for  $s \in S$  the map  $\Gamma(s)$  restricted to the domain  $S \setminus \Gamma(s)^{-1}(\theta)$  is injective. Indeed, suppose  $\Gamma(s)(m_1) = \Gamma(s)(m_2) = m$  for  $1 \leq m_1, m_2, m \leq$

$n$ . Then there exist  $g, g', h, h' \in G$ ,  $1 \leq l, l' \leq n$  such that  $s(g; m_1, l) = (g'; m, l)$  and  $s(h; m_2, l') = (h'; m, l')$ . Hence

$$(1_G; m, m)s(g; m_1, l) = (g'; m, l), \quad (1_G; m, m)s(h; m_2, l') = (h'; m, l')$$

and thus

$$(1_G; m, m)s = (x; m, m_1) = (x'; m, m_2)$$

for some  $x, x' \in G$ . It implies that  $m_1 = m_2$ , as required.

It follows that if  $\theta \notin \Gamma^n(s)(\{1, \dots, n\})$  then  $\Gamma(s)$  induces a permutation on  $\{1, \dots, n\}$ . Hence we may write  $\Gamma(s)$  in the disjoint cycle notation (cycles of length one we do not write). In the other case we may write  $\Gamma(s)$  as a product of disjoint cycles of the form  $(i_1, i_2, \dots, i_k)$  or of the form  $(i_1, i_2, \dots, i_k, \theta)$ , where  $1 \leq i_1, \dots, i_k \leq n$ . By the latter cycle we mean that  $\Gamma(s)(i_j) = i_{j+1}$  for  $1 \leq j \leq k-1$ ,  $\Gamma(s)(i_k) = \theta$ ,  $\Gamma(s)(\theta) = \theta$  and there does not exist  $1 \leq r \leq n$  such that  $\Gamma(s)(r) = i_1$ . We also agree that letters  $i, j, k$  represent elements of  $\{1, \dots, n\}$ , in other words we write explicitly  $\theta$  if the zero appears in a cycle. We agree that we do not write cycles of the form  $(i, \theta)$  in the decomposition of  $\Gamma(s)$  if  $\Gamma(s)(i) = \theta$  and  $\Gamma(s)(j) \neq i$  for every  $1 \leq j \leq n$ . If  $\Gamma(s)(i) = \theta$  for every  $1 \leq i \leq n$ , then we simply denote  $\Gamma(s)$  as  $\theta$ .

For convenience we introduce the following notation as well. If the cycle  $\varepsilon$  appears in the expression of  $\Gamma(s)$  as product of disjoint cycles then we denote this by  $\varepsilon \subseteq \Gamma(s)$ . If  $\Gamma(s)(i_1) = i'_1, \dots, \Gamma(s)(i_m) = i'_m$  then we write

$$[\dots, i_1, i'_1, \dots, i_2, i'_2, \dots, \dots, i_m, i'_m, \dots] \subseteq \Gamma(s).$$

It easily can be verified that if  $g \in G$  and  $1 \leq n_1, n_2 \leq n$  with  $n_1 \neq n_2$  then

$$(3) \quad \Gamma((g; n_1, n_2)) = (n_2, n_1, \theta) \quad \text{and} \quad \Gamma((g; n_1, n_1)) = (n_1).$$

Further, for  $s, t \in S$ ,  $g \in G$  and  $1 \leq i \leq n$ , if  $(\dots, o, m, k, \dots) \subseteq \Gamma(s)$  and  $(m_1, \dots, m_2, \theta) \subseteq \Gamma(t)$  then

$$s(g; m, i) = (g'; k, i), \quad t(g; m_2, i) = \theta,$$

for some  $g' \in G$ . Since  $s(g; o, i) = (g''; m, i)$  for some  $g'' \in G$  we obtain that

$$(g; i, m)s(g; o, i) = (g; i, m)(g''; m, i) = (gg''; i, i)$$

and thus  $(g; i, m)s(g; o, i) \neq \theta$ . Hence, there exists  $k \in G$  such that

$$(g; i, m)s = (k; i, o).$$

We claim that

$$(g; i, m_1)t = \theta.$$

Indeed, suppose this is not the case. Then  $(g; i, m_1)t = (g'; i, m_3)$  for some  $g' \in G$  and some  $m_3$ . Hence,  $(g; i, m_1)t(1; m_3, m_3) \neq \theta$  and  $t(1; m_3, m_3) \neq \theta$ . So  $\Gamma(t)(m_3) = m_1$ , a contradiction.

**Lemma 2.4.** *Let  $S$  be a finite minimal non-nilpotent semigroup with ideal  $M = \mathcal{M}^0(G, n, n; I_n)$  and  $G$  a nilpotent group. Then there exist elements  $w_1$  and  $w_2$  of  $S \setminus M$  such that*

$$(m, l) \subseteq \Gamma(w_1), (m)(l) \subseteq \Gamma(w_2)$$

or  $(\dots, m, l, m', \dots) \subseteq \Gamma(w_1), (l)(\dots, m, m', \dots) \subseteq \Gamma(w_2)$   
 or  $[\dots, k, m, \dots, l, k', \dots] \subseteq \Gamma(w_1), [\dots, l, m, \dots, k, k', \dots] \subseteq \Gamma(w_2)$   
 for pairwise distinct numbers  $l, m, m', k$  and  $k'$  between 1 and  $n$ .

*Proof.* Because of Lemma 2.1 there exists a positive integer  $m$ , distinct elements  $s_1, s_2 \in S$  and elements  $w_1, w_2, \dots, w_h \in S^1$  such that  $s_1 = \lambda_m(s_1, s_2, w_1, w_2, \dots, w_h)$ ,  $y = \rho_m(s_1, s_2, w_1, w_2, \dots, w_h)$ . Note that both  $s_1$  and  $s_2$  are non-zero. Because the semigroups  $M$  and  $S/M$  are nilpotent,  $\langle s_1, s_2, w_1, \dots, w_h \rangle \cap M \neq \emptyset$  and  $\langle s_1, s_2, w_1, \dots, w_h \rangle \cap S \setminus M \neq \emptyset$ . It follows that  $s_1, s_2 \in M$  and that there exist  $1 \leq n_1, n_2, n_3, n_4 \leq n$  and  $g, g' \in G$  such that  $s_1 = (g; n_1, n_2)$ ,  $s_2 = (g'; n_3, n_4)$ . Hence

$$[\dots, n_3, n_2, \dots, n_1, n_4, \dots] \subseteq \Gamma(w_1), [\dots, n_1, n_2, \dots, n_3, n_4, \dots] \subseteq \Gamma(w_2).$$

Here we agree that we take  $w_2 = w_1$  in case  $h = 1$ .

If  $(n_1, n_2) = (n_3, n_4)$  (for example in the case that  $h = 1$ ) then, since  $\lambda_{i-1} w_i \in (G; n_1, n_1)$  for  $1 \leq i \leq h$ , there exist  $k_i \in G$  such that  $(k; \alpha, n_2) w_i = (kk_i; \alpha, n_1)$  for every  $k \in G$  and  $\alpha \in \{n_1, n_2\}$ . Since  $\lambda_{i-1} = (g_{i-1}, n_1, n_2)$ ,  $\rho_{i-1} = (g'_{i-1}, n_1, n_2)$ , for some  $g_{i-1}, g'_{i-1} \in G$ , we get that

$$\lambda_i = (g_{i-1} k_i g'_{i-1}, n_1, n_2), \rho_i = (g'_{i-1} k_i g_{i-1}, n_1, n_2)$$

and thus

$$g = \lambda_m(g, g', k_1, \dots, k_h), g' = \rho_m(g, g', k_1, \dots, k_h).$$

Because of Lemma 2.1, this yields a contradiction with  $G$  being nilpotent. So we have shown that  $(n_1, n_2) \neq (n_3, n_4)$ . In particular, we obtain that  $h > 1$ .

We deal with two mutually exclusive cases.

(Case 1)  $n_1 = n_2 = l$ . Since  $[\dots, n_1, n_2, \dots, n_3, n_4, \dots] \subseteq \Gamma(w_2)$ , it is impossible that  $n_3 = l, n_4 \neq l$  or  $n_3 \neq l, n_4 = l$ . As  $(n_1, n_2) \neq (n_3, n_4)$  we thus obtain that  $n_3 \neq l$  and  $n_4 \neq l$ . Consequently,

$$[\dots, m, l, \dots, l, m, \dots] \subseteq \Gamma(w_1), [\dots, l, l, \dots, m, m, \dots] \subseteq \Gamma(w_2)$$

$$\text{or } [\dots, m, l, \dots, l, m', \dots] \subseteq \Gamma(w_1), [\dots, l, l, \dots, m, m', \dots] \subseteq \Gamma(w_2)$$

and thus

$$(m, l) \subseteq \Gamma(w_1), (l)(m) \subseteq \Gamma(w_2)$$

$$\text{or } (\dots, m, l, m', \dots) \subseteq \Gamma(w_1), (l)(\dots, m, m', \dots) \subseteq \Gamma(w_2)$$

for the pairwise distinct numbers  $l, m$  and  $m'$ .

(Case 2)  $n_1 \neq n_2$  and  $n_3 \neq n_4$  (the latter because otherwise, by symmetry reasons, we are as in Case 1). We obtain five possible cases:

$$(n_2 = n_3) : [\dots, m, m, \dots, l, k, \dots] \subseteq \Gamma(w_1), [\dots, l, m, \dots, m, k, \dots] \subseteq \Gamma(w_2),$$

$$(n_2 = n_4) : [\dots, k, m, \dots, l, m, \dots] \subseteq \Gamma(w_1), [\dots, l, m, \dots, k, m, \dots] \subseteq \Gamma(w_2),$$

$$(n_1 = n_3) : [\dots, m, l, \dots, m, k, \dots] \subseteq \Gamma(w_1), [\dots, m, l, \dots, m, k, \dots] \subseteq \Gamma(w_2),$$

$$(n_1 = n_4) : [\dots, k, l, \dots, m, m, \dots] \subseteq \Gamma(w_1), [\dots, m, l, \dots, k, m, \dots] \subseteq \Gamma(w_2),$$

$$[\dots, k, m, \dots, l, k', \dots] \subseteq \Gamma(w_1), [\dots, l, m, \dots, k, k', \dots] \subseteq \Gamma(w_2)$$

for some pairwise distinct psotive integers  $l, m, k, k' \leq n$ .

Cases one and four are as in the third possibility listed in the statement of the lemma. In case two we have  $\Gamma(w_1)(k) = \Gamma(w_1)(l) = m$  and  $m \neq \theta$ . If  $k \neq l$  then this contradicts with the injectivity of  $\Gamma(w_1)$  when restricted to  $\{1, \dots, n\} \setminus \Gamma(w_1)^{-1}(\theta)$ . Thus,  $k = l$  and case two is as in the third possibility listed in the statement. In case three, we have  $k = l$  (because  $\Gamma(w_1)$  is a function) and thus this case is as in the third possibility listed in the statement. Case five is one of the desired options.

Finally, because of (3) we know how the elements of  $M$  are written as products of disjoint cycles. Hence it is easily seen that  $w_1, w_2 \in (S \setminus M)$ .  $\square$

We now deal with the first case listed in Lemma 2.4.

**Lemma 2.5.** *Let  $S$  be a finite minimal non-nilpotent semigroup with ideal  $M = \mathcal{M}^0(G, n, n; I_n)$  and  $G$  a nilpotent group. Suppose there exists  $w \in S \setminus M$  such that  $(m, l) \subseteq \Gamma(w)$ . Then,*

$$S = \mathcal{M}^0(G, 2, 2; I_2) \cup \langle u \rangle,$$

the disjoint union, and

- (1)  $\langle u \rangle$  a cyclic group of order  $2^k$ ,
- (2)  $u^{2^k} = 1$  is the identity of  $S$ ,
- (3)  $\Gamma(u) = (1, 2)$  and  $\Gamma(1) = (1)(2)$ ,
- (4)  $G = \langle \Psi(u)(1), \Psi(u)(2) \rangle$ ,
- (5)  $(\Psi(u)(1) \Psi(u)(2))^{2^{k-1}} = 1$ .

A semigroup of this type we simply denote by  $U_2 = U_2(G)$ .

Furthermore,  $U_2 = \langle (g; i, j), u \rangle$  for  $(g; i, j) \in \mathcal{M}^0(G, 2, 2; I_2)$ . Note that such semigroups showed up in [5] as obstructions for a semigroup to be (PE), i.e. if a semigroup  $S$  has a semigroup of type  $U_2$  as an epimorphic image then it is not (PE).

*Proof.* Let  $e$  denote the identity of  $G$ . Obviously  $(m, l) \subseteq \Gamma(w)$  implies that  $(m)(l) \subseteq \Gamma(w^2)$ . Then because of (3), it easily is seen that

$$(4) \quad \Gamma((e; m, l)) = \lambda_2(\Gamma((e; m, l)), \Gamma((e; l, m)), \Gamma(w^2), \Gamma(w))$$

$$(5) \quad \Gamma((e; l, m)) = \rho_2(\Gamma((e; m, l)), \Gamma((e; l, m)), \Gamma(w^2), \Gamma(w)).$$

Hence the semigroup  $\langle w, (g; m, l), (g; l, m) \mid g \in G \rangle$  is not nilpotent by Lemma 2.1. Since  $S$  is minimal non-nilpotent, this implies that

$$S = \langle w, (g; m, l), (g; l, m) \mid g \in G \rangle.$$

Let  $g \in G$ . Since  $w(e; m, l) = (x; l, l)$  for some  $x \in G$ , we obtain that  $(g; m, l)w(e; m, l) = (gx; m, l) \neq \theta$ . Hence,

$$(6) \quad (g; m, l)w = (gx; m, m).$$

In particular,  $(g; m, l)w, w(g, m, l) \in I = \langle (g; m, l), (g; l, m) \mid g \in G \rangle$ . Note that  $I = \mathcal{M}^0(G, 2, 2; I_2)$ . Hence  $I$  is an ideal in the semigroup  $T = \langle w, I \rangle$ . Because of (4) and (5) the semigroup  $T$  is not nilpotent. Furthermore, for

any  $w' \in \langle w \rangle$  one easily sees that  $\Gamma(w')$  has at least two fixed points or contains a transposition in its disjoint cycle decomposition. Hence, because of (3),  $\Gamma(w') \notin \Gamma(M)$ . Therefore,  $\langle w \rangle \cap M = \emptyset$ .

Consequently, we obtain that  $S = \langle w \rangle \cup M$ , a disjoint union,  $n = 2$  and  $\Gamma(w) = (m, l)$ . It is then clear that in (4) and (5) one may replace  $w$  by  $w^{k_1}$ , with  $k_1$  an odd positive integer. It follows that the subsemigroup  $\langle w, M \rangle$  is not nilpotent. Since  $S$  is minimal non-nilpotent this implies that  $S = \langle w, M \rangle = \langle w^{k_1}, M \rangle$ . So  $w = w^r$  for some integer  $r \geq 3$ . Let  $r$  be the smallest such integer. Then  $w^{r-1}$  is an idempotent and  $\langle w \rangle$  is a cyclic group of even order. As  $\langle w \rangle = \langle w^{k_1} \rangle$  for any odd positive integer  $k_1$ , we get that  $\langle w \rangle$  has order  $2^k$  for some positive integer  $k$ .

Without loss of generality we may assume that  $m = 1$ ,  $l = 2$ . Let  $u = w$ . As  $(e; 1, 1)u^2 = (\Psi(u)(1) \Psi(u)(2); 1, 1)$  we have

$$(e; 1, 1)u^{2^k+1} = ((\Psi(u)(1) \Psi(u)(2))^{2^{k-1}} \Psi(u)(1); 1, 2).$$

Since  $\langle u \rangle$  has order  $2^k$ ,  $u^{2^k+1} = u$  and thus  $(\Psi(u)(1)\Psi(u)(2))^{2^{k-1}} = e$  and  $(e; 1, 1)u^{2^k} = (e; 1, 1)$ . It follows then easily that  $(x; 1, 1)u^{2^k} = (x; 1, 1)$  for any  $x \in G$ . Similarly one obtains that  $u^{2^k}(x; 1, 1) = (x; 1, 1)$ ,  $u^{2^k}(x; 2, 2) = (x; 2, 2)u^{2^k} = (x; 2, 2)$  for any  $x \in G$ . Hence,  $u^{2^k}$  is the identity of the semigroup  $S$ .

From (4) and (5) it easily follows that the subsemigroup

$$\mathcal{M}^0(\langle \Psi(u)(1), \Psi(u)(2) \rangle, 2, 2; I_2) \cup \langle u \rangle$$

is not nilpotent. Hence this semigroup equals  $S$  and thus  $G = \langle \Psi(u)(1), \Psi(u)(2) \rangle$ . So, indeed,  $S$  is a semigroup of type  $U_2$ .

Now suppose that  $(g; i, j) \in \mathcal{M}^0(G, 2, 2; I_2)$ . If  $i = j$  then

$$(7) \quad \Gamma((g; i, i)) = \lambda_2(\Gamma((g; i, i)), \Gamma(u(g; i, i)u), \Gamma(u), \Gamma(u^2))$$

$$(8) \quad \Gamma(u(g; i, i)u) = \rho_2(\Gamma((g; i, i)), \Gamma(u(g; i, i)u), \Gamma(u), \Gamma(u^2)).$$

Hence the semigroup  $\langle (g; i, i), u(g; i, i)u \rangle$  is not nilpotent by Lemma 2.1. Since  $S$  is minimal non-nilpotent this implies that  $S = \langle (g; i, j), u \rangle$ .

Otherwise if  $i \neq j$  we have

$$(9) \quad \Gamma((g; i, j)u) = \lambda_2(\Gamma((g; i, j)u), \Gamma(u(g; i, j)), \Gamma(u), \Gamma(u^2))$$

$$(10) \quad \Gamma(u(g; i, j)) = \rho_2(\Gamma((g; i, j)u), \Gamma(u(g; i, j)), \Gamma(u), \Gamma(u^2)).$$

Hence the semigroup  $\langle (g; i, j)u, u(g; i, j) \rangle$  is not nilpotent by Lemma 2.1. Again since  $S$  is minimal non-nilpotent this implies that  $S = \langle (g; i, j), u \rangle$ .  $\square$

Note that not every semigroup of type  $U_2$  is minimal non-nilpotent. Indeed, let  $S = \mathcal{M}^0(G, 2, 2; I_2) \cup \langle u \rangle$ , with  $\langle u \rangle$  a cyclic group of order 2,  $u^2 = 1$  is the identity of  $S$ ,  $\Gamma(u) = (1, 2)$ ,  $\Gamma(1) = (1)(2)$ ,  $\Psi(u)(1) = g$ ,  $\Psi(u)(2) = g$  and  $G$  a cyclic group  $\{1, g\}$ . The subsemigroup

$$\{(1; 1, 1), (1; 2, 2), (g; 1, 2), (g; 2, 1), u, 1, \theta\}$$



is isomorphic with  $\mathcal{M}^0(\{e\}, 2, 2; I_2) \cup \langle u \rangle$ . As it is not nilpotent and proper, the semigroup  $S$  is of type  $U_2$  but it is not minimal non-nilpotent.

In order to deal with the second case listed in Lemma 2.4 we first prove the following lemma.

**Lemma 2.6.** *Let  $S = \mathcal{M}^0(G, 3, 3; I_3) \cup \langle w_1, w_2 \rangle$  be a semigroup that is a union of the ideal  $M = \mathcal{M}^0(G, 3, 3; I_3)$  and the subsemigroup  $T = \langle w_1, w_2 \rangle$ . Suppose  $\Gamma(w_1) = (2, 1, 3, \theta)$  and  $\Gamma(w_2) = (2, 3, \theta)(1)$ . Assume  $G$  is a nilpotent group,  $\theta$  is the zero element of both  $M$  and  $S$ , and suppose  $w_2 w_1^2 = w_1^2 w_2 = w_1^3 = w_2 w_1 w_2 = \theta$ . The following properties hold.*

- (1)  $S$  is not nilpotent.
- (2)  $T$  is nilpotent.
- (3) If a subsemigroup  $S'$  of  $S$  is not nilpotent, then  $\langle w_1, w_2 \rangle \subseteq S'$ .
- (4) Every proper Rees factor semigroup of  $S$  is nilpotent.

*Proof.* (1) As

$$\Gamma((e; 1, 1)) = \lambda_2(\Gamma((e; 1, 1)), \Gamma((e; 2, 3)), \Gamma(w_1), \Gamma(w_2))$$

and

$$\Gamma((e; 2, 3)) = \rho_2(\Gamma((e; 1, 1)), \Gamma((e; 2, 3)), \Gamma(w_1), \Gamma(w_2))$$

we get that  $S$  is not nilpotent.

(2) Clearly  $I = T \setminus \langle w_2 \rangle$  is an ideal of  $T$  and  $I^3 = \{\theta\}$ . Obviously  $T/I$  is commutative and thus nilpotent. Hence,  $T$  is nilpotent.

(3) Assume  $S'$  is a subsemigroup of  $S$  that is not nilpotent. Again by Lemma 2.1, there exists a positive integer  $p$ , distinct elements  $t, t' \in S'$  and  $t_1, t_2, \dots, t_p \in S'^1$  such that  $t = \lambda_p(t, t', t_1, t_2, \dots, t_p)$ ,  $t' = \rho_p(t, t', t_1, t_2, \dots, t_p)$ . Since  $T$  is nilpotent,

$$\{t, t', t_1, t_2, \dots, t_p\} \cap \mathcal{M}^0(G, 3, 3; I_3) \neq \emptyset$$

and since  $\mathcal{M}^0(G, 3, 3; I_3)$  is an ideal of  $S$ ,  $t$  and  $t'$  are in  $\mathcal{M}^0(G, 3, 3; I_3)$ . Since  $S'$  is not nilpotent and  $\mathcal{M}^0(G, 3, 3; I_3)$  is nilpotent, we obtain that at least one of the elements  $t_1, \dots, t_p$  is in  $T$ . Now, if necessary, replacing  $t$  by  $\lambda_{i-1}(t, t', t_1, t_2, \dots, t_p)$  and  $t'$  by  $\rho_{i-1}(t, t', t_1, t_2, \dots, t_p)$ , we may assume that  $t_1 \in T$ .

Write  $t = (g_1; n_1, n_2)$  and  $t' = (g_2; n_3, n_4)$ , for some  $1 \leq n_1, n_2, n_3, n_4 \leq 3$  and  $g_1, g_2 \in G$ .

Consider the following subsets:  $A = \{x \mid \Gamma(x) = (2, 1, 3, \theta)\} = \{w_1\}$ ,  $B = \{x \mid \Gamma(x) = (2, 3, \theta)(1)\} = \{w_2\}$ ,  $C = \{x \mid \Gamma(x) = (1)\} = \{w_2^n \mid n \in \mathbb{N}, n > 1\}$ ,  $D = \{x \mid \Gamma(x) = (2, 3, \theta)\} = \{w_1 w_2^n w_1 \mid n \in \mathbb{N}\}$ ,  $E = \{x \mid \Gamma(x) = (1, 3, \theta)\} = \{w_1 w_2^n \mid n \in \mathbb{N}, n > 0\}$ ,  $F = \{x \mid \Gamma(x) = (2, 1, \theta)\} = \{w_2^n w_1 \mid n \in \mathbb{N}, n > 0\}$  and  $Z = \{x \mid \Gamma(x) = \theta\} = \{T^1 w_1^2 T^1, T^1 w_2 x_1 w_2 T^1\} \setminus \{w_1^2\}$ . By determining the images of these sets under the mapping  $\Gamma$  one sees that these sets form a partition of  $T$ . Since  $w_2 w_1^2 = w_1^2 w_2 = w_1^3 = w_2 w_1 w_2 = \theta$  we have that  $Z = \{\theta\}$ . Hence  $t_1 \notin Z$ .

If  $t_1 \in C$  then  $n_1 = n_2 = n_3 = n_4 = 1$  (because, for every  $a \in C$  we have  $\Gamma(a) = (1)$ ) and thus  $g_1 = \lambda_p(g_1, g_2, x_1, \dots, x_p)$  and  $g_2 = \rho_p(g_1, g_2, x_1, \dots, x_p)$

for some  $x_1, \dots, x_p \in G$ , in contradiction with  $G$  being nilpotent. If  $t_1 \in D$  then  $n_1 = n_3 = 2$  and  $n_2 = n_4 = 3$  (because for every  $a \in D$  we have  $\Gamma(a) = (2, 3, \theta)$ ); this again yields a contradiction with  $G$  being nilpotent. Similarity  $t_1 \notin E, F$ . Now suppose that  $t_1 = w_1$ , i.e.  $t_1 \in A$ . Since  $\Gamma(w_1) = (2, 1, 3, \theta)$ ,  $t = \lambda_p(t, t', t_1, t_2, \dots, t_p) \neq \theta$  and  $t' = \rho_p(t, t', t_1, \dots, t_p) \neq \theta$  we get that  $\{n_1, n_3\} \subseteq \{1, 2\}$ . As  $tw_1t' \neq \theta$  and  $t'w_1t \neq \theta$  we obtain that  $n_2 = \Gamma(w_1)(n_3)$  and  $n_4 = \Gamma(w_1)(n_1)$ . Hence, if  $n_1 = n_3$ , then  $n_2 = n_4$ , again yielding a contradiction with  $G$  being nilpotent. So,  $n_1 \neq n_3$ . If  $n_1 = 1$  then  $n_3 = 2$ ,  $n_4 = 3$ ,  $n_2 = 1$  and thus then  $\{(n_1, n_2), (n_3, n_4)\} = \{(1, 1), (2, 3)\}$ . Similarly, we also get the latter if  $n_1 = 2$ . It then easily can be verified that  $t_2 = w_2$  and thus  $T \subseteq S'$ , as desired. Similarly if  $t_1 = w_2$ , then  $T \subseteq S'$ .

(4) Since  $M$  is a 0-simple semigroup and because  $T$  is nilpotent, it is clear that every proper Rees factor of  $S$  is nilpotent.  $\theta$ , every proper Rees factor semigroup of  $S$  is nilpotent.  $\square$

**Lemma 2.7.** *Let  $S$  be a finite minimal non-nilpotent semigroup with ideal  $M = \mathcal{M}^0(G, n, n; I_n)$  and  $G$  a nilpotent group. Suppose there exist elements  $w_1$  and  $w_2$  of  $S \setminus M$  and pairwise distinct positive integers  $l, m, m'$  such that*

$$(\dots, m, l, m', \dots) \subseteq \Gamma(w_1) \text{ and } (l)(\dots, m, m', \dots) \subseteq \Gamma(w_2),$$

*but there do not exist distinct numbers  $l_1$  and  $l_2$  between 1 and  $n$  such that  $(l_1, l_2) \subseteq \Gamma(x)$  for some  $x \in S \setminus M$ . Then  $S$  is one of the following semigroups:*

- (1)  $U_3 = \mathcal{M}^0(G, 3, 3; I_3) \cup \langle x_1, x_2 \rangle$ , with  $\mathcal{M}^0(G, 3, 3; I_3)$  an ideal of  $U_3$ ,  $\Gamma(x_1) = (2, 1, 3, \theta)$ ,  $\Gamma(x_2) = (2, 3, \theta)(1)$ ,  $x_2x_1^2 = x_1^2x_2 = x_1^3 = x_2x_1x_2 = \theta$  (the zero element of  $S$ ),

$$G = \langle \Psi(x_1)(1), \Psi(x_1)(2), \Psi(x_2)(1), \Psi(x_2)(2) \rangle;$$

- (2)  $U_4 = \mathcal{M}^0(G, n, n; I_n) \cup \langle v_1, v_2 \rangle$ , with  $\mathcal{M}^0(G, n, n; I_n)$  an ideal of  $U_4$ ,

$$[\dots, k_1, k_2, \dots, k_3, k_4, \dots] \subseteq \Gamma(v_1), [\dots, k_1, k_4, \dots, k_3, k_2, \dots] \subseteq \Gamma(v_2)$$

*for pairwise distinct numbers  $k_1, k_2, k_3$  and  $k_4$  between 1 and  $n$ ,  $G = \langle \Psi(v_1)(1), \dots, \Psi(v_1)(n), \Psi(v_2)(1), \dots, \Psi(v_2)(n), \theta \rangle \setminus \{\theta\}$  and there do not exist pairwise distinct numbers  $o_1, o_2$  and  $o_3$  between 1 and  $n$  such that  $(o_2, o_1, o_3, \theta) \subseteq \Gamma(y)$ ,  $(o_2, o_3, \theta)(o_1) \subseteq \Gamma(z)$  for some  $y, z \in \langle v_1, v_2 \rangle$ .*

*Furthermore, if  $U_3$  is minimal non-nilpotent then  $U_3 = \langle (g; 1, 1), (g'; 2, 3), x_1, x_2 \rangle$  and if  $U_4$  is minimal non-nilpotent then  $U_4 = \langle (g; k_1, k_4), (g'; k_3, k_2), v_1, v_2 \rangle$ , for every  $g, g' \in G$ .*

*Proof.* Note that, by Lemma 2.5, the assumption that there do not exist distinct numbers  $l_1$  and  $l_2$  between 1 and  $n$  such that  $(l_1, l_2) \subseteq \Gamma(x)$  for some  $x \in S \setminus M$ , excludes that  $S$  is a semigroup of type  $U_2$ .

First suppose that there exist integers  $o_1, o_2, o_3$  between 1 and  $n$  and elements  $x_1, x_2 \in S$  such that  $(o_2, o_1, o_3, \theta) \subseteq \Gamma(x_1)$  and  $(o_2, o_3, \theta)(o_1) \subseteq \Gamma(x_2)$ . It is then readily verified that  $\mathcal{M}^0(G, \{o_1, o_2, o_3\}, \{o_1, o_2, o_3\}; I_{\{o_1, o_2, o_3\}})$  is

an ideal in the subsemigroup  $\mathcal{M}^0(G, \{o_1, o_2, o_3\}, \{o_1, o_2, o_3\}; I_{\{o_1, o_2, o_3\}}) \cup \langle x_1, x_2 \rangle$  of  $S$ . Because

$$(11) \quad \Gamma((e; o_1, o_1)) = \lambda_2(\Gamma((e; o_1, o_1)), \Gamma((e; o_2, o_3)), \Gamma(x_1), \Gamma(x_2)),$$

$$(12) \quad \Gamma((e; o_2, o_3)) = \rho_2(\Gamma((e; o_1, o_1)), \Gamma((e; o_2, o_3)), \Gamma(x_1), \Gamma(x_2))$$

we get that the semigroup  $\mathcal{M}^0(G, \{o_1, o_2, o_3\}, \{o_1, o_2, o_3\}; I_{\{o_1, o_2, o_3\}}) \cup \langle x_1, x_2 \rangle$  is not nilpotent and thus, as  $S$  is minimal non-nilpotent,  $S = \mathcal{M}^0(G, 3, 3; I_3) \cup \langle x_1, x_2 \rangle$ .

We now show that  $x_1$  and  $x_2$  satisfy the required relations. Let  $T = \langle x_1, x_2 \rangle$ . Consider the following subsets:  $A = \{x_1\}$ ,  $B = \{x_2\}$ ,  $C = \{x_2^n \mid n \in \mathbb{N}, n > 1\}$ ,  $D = \{x_1 x_2^n x_1 \mid n \in \mathbb{N}\}$ ,  $E = \{x_1 x_2^n \mid n \in \mathbb{N}, n > 0\}$ ,  $F = \{x_2^n x_1 \mid n \in \mathbb{N}, n > 0\}$  and  $Z = \{T^1 x_1^2 T^1, T^1 x_2 x_1 x_2 T^1\} \setminus \{x_1^2\}$ . By determining the images of these sets under the mapping  $\Gamma$  one sees that these sets form a partition of  $T$ . Since  $S' = \{T^1 x_1^2 T^1, T^1 x_2 x_1 x_2 T^1\} \setminus \{x_1^2\} \cup \{\theta\}$  is an ideal of  $S$ , it easily follows from (11) and (12) that  $S/S'$  is non-nilpotent. Hence,  $S' = \{\theta\}$  and thus  $Z = \{\theta\}$  and thus  $x_1^3 = x_1^2 x_2 = x_2 x_1^2 = x_2 x_1 x_2 = \theta$ , as desired. From Lemma 2.6.(2) we know that the subsemigroup  $T = \langle x_1, x_2 \rangle$  is nilpotent.

From (11) and (12) it easily can be verified that the subsemigroup

$$\mathcal{M}^0(\langle \Psi(w_1)(o_1), \Psi(w_1)(o_2), \Psi(w_2)(o_1), \Psi(w_2)(o_2) \rangle, 3, 3; I_3) \cup \langle w_1, w_2 \rangle$$

is not nilpotent. We therefore obtain that

$$G = \langle \Psi(x_1)(o_1), \Psi(x_1)(o_2), \Psi(x_2)(o_1), \Psi(x_2)(o_2) \rangle.$$

So, this finishes the proof of the fact that  $S$  is a semigroup of type  $U_3$ .

Furthermore, for  $g, g' \in G$ , since

$$\Gamma((g; o_1, o_1)) = \lambda_2(\Gamma((g; o_1, o_1)), \Gamma((g'; o_2, o_3)), \Gamma(x_1), \Gamma(x_2))$$

and

$$\Gamma((g'; o_2, o_3)) = \rho_2(\Gamma((g; o_1, o_1)), \Gamma((g'; o_2, o_3)), \Gamma(x_1), \Gamma(x_2)),$$

we get that the subsemigroup  $\langle (g; o_1, o_1), (g'; o_2, o_3), x_1, x_2 \rangle$  is not nilpotent. So, if  $S$  is minimal non-nilpotent then this implies that  $S = \langle (g; o_1, o_1), (g'; o_2, o_3), x_1, x_2 \rangle$ .

In the remainder of the proof we assume that here do not exist pairwise distinct numbers  $o_1, o_2$  and  $o_3$  between 1 and  $n$  such that  $(o_2, o_1, o_3, \theta) \subseteq \Gamma(y)$ ,  $(o_2, o_3, \theta)(o_1) \subseteq \Gamma(z)$  for some  $y, z \in S$ . Further, without loss of generality we may assume that  $m = 1$ ,  $l = 3$  and  $m' = 2$ . So  $(\dots, 1, 3, 2, \dots) \subseteq \Gamma(w_1)$  and  $(3) (\dots, 1, 2, \dots) \subseteq \Gamma(w_2)$ .

If  $\Gamma(w_1)(2) = r \neq \theta$  then  $[\dots, 1, 2, \dots, 3, r, \dots] \subseteq \Gamma(w_1^2)$ ,  $[\dots, 1, r, \dots, 3, 2, \dots] \subseteq \Gamma(w_1 w_2)$ . We get that

$$\Gamma((e; 1, r)) = \lambda_2(\Gamma((e; 1, r)), \Gamma((e; 3, 2)), \Gamma(w_1^2), \Gamma(w_1 w_2))$$

and

$$\Gamma((e; 3, 2)) = \rho_2(\Gamma((e; 1, r)), \Gamma((e; 3, 2)), \Gamma(w_1^2), \Gamma(w_1 w_2)).$$

Let  $v_1 = w_1^2$  and  $v_2 = w_1 w_2$ . Then we obtain that the subsemigroup  $\mathcal{M}^0(G, n, n; I_n) \cup \langle v_1, v_2 \rangle$  is not nilpotent. As  $S$  is minimal non-nilpotent this yields that  $S = \mathcal{M}^0(G, n, n; I_n) \cup \langle v_1, v_2 \rangle$ , a semigroup of type  $U_4$ . Furthermore, It easily is verified that

$$\Gamma((g; 1, r)) = \lambda_2(\Gamma((g; 1, r)), \Gamma((g'; 3, 2)), \Gamma(v_1), \Gamma(v_2))$$

and

$$\Gamma((g'; 3, 2)) = \rho_2(\Gamma((g; 1, r)), \Gamma((g'; 3, 2)), \Gamma(v_1), \Gamma(v_2)).$$

Hence, for any  $g, g' \in G$ , the subsemigroup  $\langle (g; 1, r), (g'; 3, 2), v_1, v_2 \rangle$  is not nilpotent. Therefore,  $U_4 = \langle (g; 1, r), (g'; 3, 2), v_1, v_2 \rangle$  for every  $g, g' \in G$ .

In the remainder of the proof we may thus also assume that  $\Gamma(w_1)(2) = \theta$ . Next we show that we also may assume that there does not exist an integer  $r'$ , with  $1 \leq r' \leq n$ , such that  $\Gamma(w_1)(r') = 1$ . Indeed, suppose the contrary and let  $r'$  be such an integer. Then  $[\dots, 1, 2, \dots, r', 3, \dots] \subseteq \Gamma(w_1^2)$ ,  $[\dots, 1, 3, \dots, r', 2, \dots] \subseteq \Gamma(w_2 w_1)$  and thus  $S$  is a semigroup of type  $U_4$ . This proves the claim and thus  $(1, 3, 2, \theta) \subseteq \Gamma(w_1)$ .

We now show that this leads to a contradiction. We claim that the cycle  $(\dots, 1, 2, \dots)$  in  $\Gamma(w_2)$  ends in  $\theta$ . Indeed, assume the contrary. That is, this cycle ends in a positive integer. Let  $n_2$  denote the length of this cycle. Then,  $(1, 3) \subseteq \Gamma(w_2^{n_2-1} w_1)$ ,  $(1)(3) \subseteq \Gamma(w_2^{n_2})$ . However, as mentioned above, this is impossible. This proves the claim and thus there exist positive integers  $k, k'$  and  $n'$  such that  $\Gamma(w_2)(k) = k' \neq \theta$ ,  $\Gamma(w_2)(k') = \theta$  and  $\Gamma(w_2^{n'})(1) = k'$ . So  $(3)(\dots, 1, 2, \dots, k, k', \theta) \subseteq \Gamma(w_2)$ .

If  $\Gamma(w_2^{n'-1} w_1)(k') = k'' \neq \theta$  then  $[\dots, 1, k', \dots, 3, k'', \dots] \subseteq \Gamma((w_2^{n'-1} w_1)^2)$ ,  $[\dots, 1, k'', \dots, 3, k', \dots] \subseteq \Gamma(w_2^{n'-1} w_1 w_2^{n'})$ . Since  $\Gamma(w_2)(3) = 3$ ,  $\Gamma(w_2)(1) = 2$  and  $\Gamma(w_2)(k') = \theta$ , it is clear that  $1, k'$  and  $3$  are pairwise distinct. As  $\Gamma(w_2^{n'-1} w_1)(k') = k'' \neq \theta$ ,  $\Gamma(w_1)(k') = \alpha \neq \theta$ . As  $(1, 3, 2, \theta) \subseteq \Gamma(w_1)$  and because  $1, k', 3$  are pairwise distinct integers, we get that  $\alpha \notin \{1, 2, 3\}$ . We claim that  $1, k', 3$  and  $k''$  are pairwise distinct and thus that  $S$  is a semigroup of type  $U_4$ . Indeed, for otherwise,  $k'' = 3$  or  $k'' = k'$  or  $k'' = 1$ . The former is excluded as it implies that  $\alpha = 3$ . If  $k'' = k'$ , then  $\alpha = 2$ , a contradiction. If  $k'' = 1$  then  $(\alpha, 3) \subseteq \Gamma(w_1 w_2^{n'-1} w_1 w_2^{2n'-1})$  and thus  $S \cong U_2$ , again a contradiction.

If  $\Gamma(w_2^{n'-1} w_1)(k') = \theta$ , we get that  $(1, 3, k', \theta) \subseteq \Gamma(w_2^{n'-1} w_1)$ . Clearly,  $(3)(1, k', \theta) \subseteq \Gamma(w_2^{n'})$ . However, this yields a contradiction with our assumptions.

Finally, consider a semigroup  $U_4$  that is minimal non-nilpotent (with notations as in the statement). Let

$$H_1 = \langle \Psi(v_1)(1), \dots, \Psi(v_1)(n), \Psi(v_2)(1), \dots, \Psi(v_2)(n), \theta \rangle,$$

the semigroup generated by all listed elements. Note that  $H = H_1 \setminus \{\theta\}$  is a subgroup of the maximal subgroup  $G$  defining  $M$ . Since

$$\begin{aligned} \Gamma((e; k_3, k_2)) &= \lambda_2(\Gamma((e; k_3, k_2)), \Gamma((e; k_1, k_4)), \Gamma(v_1), \Gamma(v_2)) \\ \Gamma((e; k_1, k_4)) &= \rho_2(\Gamma((e; k_3, k_2)), \Gamma((e; k_1, k_4)), \Gamma(v_1), \Gamma(v_2)) \end{aligned}$$

we get that  $\mathcal{M}^0(H, n, n, I_n) \cup \langle v_1, v_2 \rangle$  is not nilpotent. Because, by assumption,  $U_4$  is minimal non-nilpotent, we obtain that

$$\mathcal{M}^0(H, n, n, I_n) \cup \langle v_1, v_2 \rangle = \mathcal{M}^0(G, n, n, I_n) \cup \langle v_1, v_2 \rangle.$$

We now show that  $G = H$ . Suppose the contrary, then there exists  $g \in G \setminus H$ . Let  $\alpha = (g; 1, 1)$ . Clearly,  $\alpha \notin \mathcal{M}^0(H, n, n, I_n)$  and thus  $\alpha \in \langle v_1, v_2 \rangle$ . Since  $(g; 1, 1)(1; 1, 1) \neq \theta$  we get that  $\Gamma(\alpha)(1), \Psi(\alpha)(1) \neq \theta$  and  $(g; 1, 1) = (g; 1, 1)(1; 1, 1) = \alpha(1; 1, 1) = (\Psi(\alpha)(1); \Gamma(\alpha)(1), 1)$  and thus  $g = \Psi(\alpha)(1)$ . This contradicts with  $g \notin H$ . So, indeed,  $G = H$ .

Also, as before one easily verifies that if  $U_4$  is minimal non-nilpotent then, for every  $g, g' \in G$ , we get that  $U_4 = \langle (g; k_1, k_4), (g'; k_3, k_2), v_1, v_2 \rangle$ .  $\square$

Note that if the semigroup  $S = \mathcal{M}^0(G, n, n, I_n) \cup \langle v_1, v_2 \rangle$  is minimal non-nilpotent of type  $U_4$  then, in general, it does not imply that  $\langle v_1, v_2 \rangle$  is nilpotent. Let  $S = \mathcal{M}^0(\{e\}, 4, 4, I_4) \cup \langle v_1, v_2 \rangle$  with  $v_1 = (1, 2, 3, 4), v_2 = (1, 4, 3, 2, \theta)$ . For example, the semigroup  $S$  is minimal non-nilpotent and of type  $U_4$ . One can verify that  $S = \langle v_1, v_2 \rangle$  and thus the subsemigroup  $\langle v_1, v_2 \rangle$  is not nilpotent.

We can now state our main result.

Our main result now follows at once from Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.7 (the second part in the enumeration follows from the fact that  $\Gamma^{-1}(\theta)$  is an ideal of  $S$  and  $S/\Gamma^{-1}(\theta)$  is not nilpotent).

**Theorem 2.8.** *Let  $S$  be a finite minimal non-nilpotent semigroup. Then  $S$  is either a Schmidt group or a semigroup of type  $U_1, U_2, U_3$  or  $U_4$ . In particular, the semigroups  $U_n$ , with  $2 \leq n \leq 4$ , are generated by four elements, they have a two-generated subsemigroup  $T$  and an ideal  $M = \mathcal{M}^0(G; n, n, I_n)$  (with  $G$  a nilpotent group) such that*

$$S = M \cup T,$$

and there exists a representation  $\Gamma$  of  $S$  to the full transformation semigroup on  $\{1, \dots, n\} \cup \{\theta\}$  such that, for all  $s \in S$ ,

- (1)  $\Gamma(s)(\theta) = \theta$ ,
- (2)  $\Gamma(s)$  is injective when restricted to  $\{1, \dots, n\} \setminus \Gamma(s)^{-1}(\theta)$ ,
- (3)  $|\Gamma^{-1}(\theta)| = 1$ ,
- (4) if  $T$  has a zero element, say  $\theta_T$ , then  $\theta_T = \theta$  (the zero of  $S$ ).

Furthermore,  $\Gamma(S)$  also is a minimal non-nilpotent semigroup.

*Proof.* All parts, except the items (3) and (4), follow at once from Lemma 2.3, Lemma 2.4, Lemma 2.5 and Lemma 2.7. Part (3) follows from the fact that  $\Gamma^{-1}(\theta)$  is an ideal of  $S$  and  $S/\Gamma^{-1}(\theta)$  is not nilpotent.

To prove part (4), assume  $T$  has a zero element, say  $\theta_T$ . We prove by contradiction that  $\theta_T = \theta$ . So suppose that  $\theta_T \neq \theta$ . Then, by part (3),  $\Gamma(\theta_T) \neq \theta$ . Hence, there exists  $i$  between 1 and  $n$  such that  $\Gamma(\theta_T)(i) \neq \theta$ .

Now let  $t \in T$ . We have

$$\theta_T(1; i, i) = (\Psi(\theta_T)(i); \Gamma(\theta_T)(i), i)$$

and

$$\begin{aligned} \theta_T t(1; i, i) &= \theta_T(\Psi(t)(i); \Gamma(t)(i), i) \\ &= (\Psi(\theta_T)(\Gamma(t)(i))\Psi(t)(i); \Gamma(\theta_T)(\Gamma(t)(i)), i). \end{aligned}$$

Because  $\theta_T t = \theta_T$  we obtain that  $\Gamma(\theta_T)(\Gamma(t)(i)) = \Gamma(\theta_T)(i)$ . Now as  $\Gamma(\theta_T)(i) \neq \theta$ ,  $\Gamma(t)(i) = i$ . Therefore, for every  $t \in T$ , we have  $(i) \subseteq \Gamma(t)$ . Because  $\Gamma((g; \alpha, \beta)) = (\beta, \alpha, \theta)$  for every  $(g; \alpha, \beta) \in M$ , it follows that  $M \cap T = \emptyset$ . Let  $M' = \mathcal{M}^0(G, \{1, \dots, i-1, i+1, \dots, n\}, \{1, \dots, i-1, i+1, \dots, n\}; I_{n-1})$ . Since  $(i) \subseteq \Gamma(t)$  and  $\Gamma(t)$  restricted to  $\{1, \dots, n\} \setminus \Gamma(t)^{-1}(\theta)$  is injective for every  $t \in T$ , we get that  $M'T, TM' \subseteq M'$  and thus  $M' \cup T$  is a subsemigroup of  $S$ . Lemma 2.4 implies that there exist elements  $w_1$  and  $w_2$  of  $T$  such that

$$(m, l) \subseteq \Gamma(w_1), (m)(l) \subseteq \Gamma(w_2)$$

$$\text{or } (\dots, m, l, m', \dots) \subseteq \Gamma(w_1), (l)(\dots, m, m', \dots) \subseteq \Gamma(w_2)$$

$$\text{or } [\dots, k, m, \dots, l, k', \dots] \subseteq \Gamma(w_1), [\dots, l, m, \dots, k, k', \dots] \subseteq \Gamma(w_2)$$

for pairwise distinct numbers  $l, m, m', k$  and  $k'$  between 1 and  $n$ . As  $\Gamma(w_1)(o) \neq o$  for  $o \in \{l, m, m', k, k'\}$ ,  $i \notin \{l, m, m', k, k'\}$ . Hence  $M' \cup T$  is not nilpotent. As  $T \cap M = \emptyset$  and  $M \neq M'$ ,  $S \neq M' \cup T$  and it is a contradiction with  $S$  is minimal non-nilpotent.  $\square$

Note that in general the subsemigroups  $T$  and  $M$  of a minimal non-nilpotent semigroup  $U_n$  (listed in Theorem 2.8) are not  $\theta$ -disjoint ( $\theta$  disjoint means that if there is a common element then it is  $\theta$ ). We now show that  $U_n$  (with  $2 \leq n \leq 4$ ) is an epimorphic image of a semigroup built on  $\theta$ -disjoint semigroups.

Let  $T$  be a semigroup with the zero  $\theta_T$  and  $M$  be a nilpotent regular Rees matrix semigroup  $\mathcal{M}^0(G, n, n; I_n)$ . Let  $\Gamma$  be a representation of  $T$  to the full transformation semigroup  $\mathcal{T}_{\{1, \dots, n\} \cup \{\theta\}}$  such that for every  $t \in T$ ,  $\Gamma(t)(\theta) = \theta$ ,  $|\Gamma^{-1}(\theta)| \leq 1$  (as agreed before, by  $\theta$  we also denote the constant map onto  $\theta$ ),  $\Gamma(t)$  restricted to  $\{1, \dots, n\} \setminus \Gamma(t)^{-1}(\theta)$  is injective and  $\Gamma(\theta_T) = \theta$ . Further, for every  $t \in T$ , let

$$\Psi(t) : \{1, \dots, n\} \cup \{\theta\} \rightarrow G^\theta$$

be a map (as considered in (2)) such that  $\Psi(t)(i) \neq \theta$  if and only if  $\Gamma(t)(i) \neq \theta$  and  $\Psi(t_1 t_2) = (\Psi(t_1) \circ \Gamma(t_2)) \Psi(t_2)$  for every  $t_1, t_2 \in T$ .

We define a semigroup denoted by

$$S = \mathcal{M}^0(G, n, n; I_n) \cup_{\Psi}^{\Gamma} T.$$

As sets this is the  $\theta$ -disjoint union of  $\mathcal{M}^0(G, n, n; I_n)$  and  $T$  (i.e. the disjoint union with the zeros identified). The multiplication is such that  $T$  and  $M$  are subsemigroups,

$$t(g; i, j) = \begin{cases} (\Psi(t)(i)g; \Gamma(t)(i), j) & \text{if } \Gamma(t)(i) \neq \theta \\ \theta & \text{otherwise} \end{cases}$$

and

$$(g; i, j)t = \begin{cases} (g\Psi(t)(j'); i, j') & \text{if } \Gamma(t)(j') = j \\ \theta & \text{otherwise} \end{cases}$$

It easily can be verified that  $S$  is associative.

Note that if  $G = \{e\}$ , then  $\Psi(t)(i) = e$  if and only if  $\Gamma(t)(i) \neq \theta$ . In this case we denote  $\Psi$  simply as  $id$ .

It follows from Theorem 2.8 (and its proof) that the minimal non-nilpotent semigroups  $U_n$  (with  $2 \leq n \leq 4$ ) are an epimorphic image of a semigroup of the type  $\mathcal{M}^0(G, n, n; I_n) \cup_{\Psi}^{\Gamma} T$ , with  $G$  a nilpotent group and  $T$  a two-generated nilpotent semigroup with a zero.

**Corollary 2.9.** *Every finite minimal non-nilpotent semigroup  $S$  is an epimorphic image of one of the following semigroups:*

- (1) a Schmidt group,
- (2)  $U_1 = \{e, f\}$  with  $e^2 = e$ ,  $f^2 = f$ ,  $ef = f$  and  $fe = e$ ,
- (3)  $\mathcal{M}^0(G, 2, 2; I_2) \cup_{\Psi}^{\Gamma} T$  such that  $T = \langle u \rangle \cup \{\theta\}$  with  $\theta$  the zero of  $S$ ,  $u^{2^k} = 1$  the identity of  $T \setminus \{\theta\}$  (and of  $S$ ) and  $\Gamma(u) = (1, 2)$ .
- (4)  $\mathcal{M}^0(G, 3, 3; I_3) \cup_{\Psi}^{\Gamma} \langle w_1, w_2 \rangle$ , with  $\Gamma(w_1) = (2, 1, 3, \theta)$  and  $\Gamma(w_2) = (2, 3, \theta)(1)$ ,  $w_2 w_1^2 = w_1^2 w_2 = w_1^3 = w_2 w_1 w_2 = \theta$ .
- (5)  $\mathcal{M}^0(G, n, n; I_n) \cup_{\Psi}^{\Gamma} \langle v_1, v_2 \rangle$ , with

$$[\dots, k, m, \dots, k', m', \dots] \in \Gamma(v_1), [\dots, k, m', \dots, k', m, \dots] \in \Gamma(v_2)$$

for pairwise distinct numbers  $k, k', m$  and  $m'$  between 1 and  $n$ , there do not exist distinct numbers  $l_1$  and  $l_2$  between 1 and  $n$  such that  $(l_1, l_2) \in \Gamma(x)$  for some  $x \in \langle v_1, v_2 \rangle$  and there do not exist pairwise distinct numbers  $o_1, o_2$  and  $o_3$  between 1 and  $n$  such that  $(o_2, o_1, o_3, \theta) \in \Gamma(y_1)$ ,  $(o_2, o_3, \theta)(o_1) \in \Gamma(y_2)$  for some  $y_1, y_2 \in \langle v_1, v_2 \rangle$ .

Note that semigroups of type  $U_3$  and  $U_4$  are not necessarily minimal non-nilpotent. For example, it is readily verified that there exists semigroups

$$S = \mathcal{M}^0(\{1, g\}, 3, 3; I_3) \cup_{id}^{\Gamma} \langle w, v \rangle$$

such that  $v^2 = v^3$ ,  $wv^2 = wv$ ,  $vw = v^2w$ ,  $w^2 = wvw = wv^2w$ ,  $vw^2 = w^2v = w^3 = v w v = \theta$ ,  $\Gamma(w) = (2, 1, 3, \theta)$  and  $\Gamma(v) = (2, 3, \theta)(1)$ . Such a semigroup  $S$  has  $\mathcal{M}^0(\{e\}, 3, 3; I_3) \cup_{id}^{\Gamma} \langle w, v \rangle$  as a proper semigroup. The latter is minimal non-nilpotent and thus  $S$  is not minimal non-nilpotent.

We now give several concrete examples. The first example shows that for a minimal non-nilpotent semigroup  $U_4$  the maximal subgroup of  $M$  does not have to be trivial.

Let

$$S = \mathcal{M}^0(G, 4, 4; I_4) \cup_{\Psi}^{\Gamma} \langle w, v \rangle$$

with  $G = \{1, g\}$  is a cyclic group,  $w^2 = v^2 = wv = vw = \theta$ ,

$$\Gamma(w) = (4, 1, \theta)(3, 2, \theta) \text{ and } \Gamma(v) = (4, 2, \theta)(3, 1, \theta),$$

$$\Psi(w)(4) = \Psi(w)(3) = \Psi(v)(4) = 1, \quad \Psi(v)(3) = g \text{ and } \langle w, v \rangle = \{w, v, \theta\}.$$



Since

$$\Gamma((1_G; 3, 1)) = \lambda_2(\Gamma((1_G; 3, 1)), \Gamma((1_G; 4, 2)), \Gamma(w), \Gamma(v))$$

and

$$\Gamma((1_G; 4, 2)) = \rho_2(\Gamma((1_G; 3, 1)), \Gamma((1_G; 4, 2)), \Gamma(w), \Gamma(v)),$$

we obtain that  $S$  is not nilpotent. Suppose that a subsemigroup  $S'$  of  $S$  is not nilpotent. By Lemma 2.1, there exists a positive integer  $m$ , distinct elements  $x, y \in S'$  and elements  $w_1, w_2, \dots, w_m \in S'^1$  such that  $x = \lambda_m(x, y, w_1, w_2, \dots, w_m)$ ,  $y = \rho_m(x, y, w_1, w_2, \dots, w_m)$ . Since  $\langle w, v \rangle$  is nilpotent,

$$\{x, y, w_1, w_2, \dots, w_m\} \cap \mathcal{M}^0(G, 4, 4; I_4) \neq \emptyset$$

and since  $\mathcal{M}^0(G, 4, 4; I_4)$  is an ideal of  $S$ ,  $x$  and  $y$  are non-zero elements in  $\mathcal{M}^0(G, 4, 4; I_4)$ . Since  $S'$  is not nilpotent and  $\mathcal{M}^0(G, 4, 4; I_4)$  is nilpotent, at least one element of the set  $\{w_1, \dots, w_m\}$  is not in  $\mathcal{M}^0(G, 4, 4; I_4)$ . As before, without loss of generality, we may suppose that  $w_1 \notin \mathcal{M}^0(G, 4, 4; I_4)$ . Write  $x = (g_1; n_1, n_2)$  and  $y = (g_2; n_3, n_4)$ , for some  $1 \leq n_1, n_2, n_3, n_4 \leq 4$  and  $g_1, g_2 \in G$ .

If  $w_1 = w$  then  $\{(n_1, n_2), (n_3, n_4)\} = \{(3, 1), (4, 2)\}$  and it easily can be verified that  $w_2 = v$ .

It easily is verified that  $\mathcal{M}^0(\{e\}, 4, 4; I_4)$  is a subsemigroup of  $\langle \Gamma((g_1; 3, 1)), \Gamma((g_2; 4, 2)), \Gamma(w), \Gamma(v) \rangle$ . It follows that  $\mathcal{M}^0(\{e\}, 4, 4; I_4) \subseteq \Gamma(S')$ . Hence, for every  $1 \leq \alpha, \beta \leq 4$ , there exists an element  $p \in S'$  such that  $\Gamma(p) = (\beta, \alpha, \theta)$ . As  $\Gamma(w) = (4, 1, \theta)(3, 2, \theta)$  and  $\Gamma(v) = (4, 2, \theta)(3, 1, \theta)$  and  $\langle w, v \rangle = \{w, v, \theta\}$ , we obtain that there exists an element  $h \in \{1, g\}$  such that  $p = (h; \alpha, \beta)$ .

If  $S \neq S'$  then there exists an element  $(k; i, j) \in \mathcal{M}^0(G, 4, 4; I_4)$  such that  $(k; i, j) \notin S'$ . Now suppose that  $(n_1, n_2) = (3, 1)$  and  $(n_3, n_4) = (4, 2)$ . Then

$$v(g_1; 3, 1)v = (gg_1g; 1, 3), \quad w(g_2; 4, 2)w = (g_2; 1, 3),$$

$$v(g_1; 3, 1)w = (gg_1; 1, 4), \quad w(g_2; 4, 2)v = (g_2; 1, 4).$$

Since  $g_1, g_2 \in \{1, g\}$  we get that both  $(1; 1, 3)$  and  $(g; 1, 3)$ , or both  $(1; 1, 4)$  and  $(g; 1, 4)$  are in  $S'$ . Suppose that both  $(1; 1, 3)$  and  $(g; 1, 3)$  are in  $S'$ . As proved above, there exist elements  $k_1, k_2 \in G$  such that  $(k_1; i, 1), (k_2; 3, j) \in S'$ . Then  $(k_1; i, 1)(1; 1, 3)(k_2; 3, j), (k_1; i, 1)(g; 1, 3)(k_2; 3, j) \in S'$  and thus  $(k_1k_2; i, j), (k_1gk_2; i, j) \in S'$ . Since  $k, k_1, k_2 \in \{1, g\}$  we get that  $(k; i, j)$  is in  $S'$ , a contradiction. So,  $S = S'$  in this case.

Similarly,  $(1; 1, 4), (g; 1, 4) \in S'$  leads to  $S = S'$ . Hence, we have proved that  $S = S'$  if  $w_1 = w$ . If  $w_1 = v$ , then one proves in an analogous manner that  $S = S'$ . So, indeed,  $S$  is minimal non-nilpotent semigroup of type  $U_4$ .

We finish with constructing examples of minimal non-nilpotent semigroups, denoted  $Y_n$ , of type  $U_4$  with a nilpotent ideal  $\mathcal{M}^0(\{e\}, n, n; I_n)$  for arbitrary  $n \geq 5$ . Let  $n \geq 5$  and consider  $\mathcal{M}^0(\{e\}, n, n; I_n)$  as a subsemigroup of the full transformation semigroup (see (3)) on  $\{1, \dots, n\} \cup \{\theta\}$ , i.e. we



identify  $(e; i, j)$  with the cycle  $(j, i, \theta)$  if  $i \neq j$  and  $(e, i, i)$  with the permutation  $(i)$ . Let

$$Y_n = \mathcal{M}^0(\{e\}, n, n; I_n) \cup_{id}^\Gamma \langle w, v \rangle$$

and

$$\Gamma(w) = (2, 3, \theta)(4, 1, \theta), \quad \Gamma(v) = (2, 1, \theta)(n, n-1, \dots, 6, 5, 4, 3, \theta).$$

It easily can be verified that

$$\Gamma(v^p w^q) = \Gamma(w^k) = \Gamma(v^l) = \theta \text{ for } p, q \geq 1, k \geq 2, l \geq n-2$$

$$\Gamma(w^q v^p) = \theta \text{ for } q \geq 2, p \geq 1,$$

$$\Gamma(w v^p) = (p+4, 1, \theta) \text{ for } n-4 \geq p \geq 1, \Gamma(w v^p) = \theta \text{ for } p > n-4 \text{ and}$$

$$\Gamma(a w v^p) = \theta \text{ for } p \geq 1, a \in \langle w, v \rangle.$$

Because  $|\Gamma^{-1}(\theta)| = 1$  we thus obtain that  $v^p w^q = w^k = v^l = \theta$  for  $p, q \geq 1, k \geq 2, l \geq n-2$ ,  $w^q v^p = \theta$  for  $q \geq 2, p \geq 1$  and  $a w v^p = \theta$  for  $p \geq 1, a \in \langle w, v \rangle$ . Thus

$$\langle w, v \rangle = \{w, v, \dots, v^{n-3}, wv, \dots, wv^{n-4}, \theta\}$$

and clearly  $\langle w, v \rangle^n = \{\theta\}$ . Therefore  $\langle w, v \rangle$  is nilpotent.

We claim that  $Y_n$  is minimal non-nilpotent. To prove this, suppose that  $Y$  is a subsemigroup of  $Y_n$  that is not nilpotent. We need to prove that  $Y = Y_n$ . As before, there exists a positive integer  $m$ , distinct elements  $x, y \in \mathcal{M}^0(\{e\}, n, n; I_n)$  and elements  $w_1, w_2, \dots, w_m \in Y^1$  with  $w_1 \notin \mathcal{M}^0(\{e\}, n, n; I_n)$  such that  $x = \lambda_m(x, y, w_1, w_2, \dots, w_m)$ ,  $y = \rho_m(x, y, w_1, w_2, \dots, w_m)$ . Write  $x = (e; n_1, n_2)$ ,  $y = (e; n_3, n_4)$  for some  $1 \leq n_1, n_2, n_3, n_4 \leq n$ . Since  $x \neq y$ ,  $(n_1, n_2) \neq (n_3, n_4)$ .

Since  $\Gamma(x w_1 y)$  and  $\Gamma(y w_1 x)$  are non-zero,  $\Gamma(w_1)(n_3) = n_2$  and  $\Gamma(w_1)(n_1) = n_4$ . Now as  $\Gamma(w v^p) = (p+4, 1, \theta)$  for  $n-4 \geq p \geq 1$  and  $(n_1, n_2) \neq (n_3, n_4)$ ,  $w_1 \notin \{wv, \dots, wv^{n-4}\}$ . Similarly  $w_2 \notin \{wv, \dots, wv^{n-4}\}$ . So,  $w_1 = w$  or  $w_1 = v^k$  for some  $k \geq 1$ .

Suppose that  $w_1 = v^k$ . Since  $\Gamma(x w_1 y)$  and  $\Gamma(y w_1 x)$  are non-zero,  $\Gamma(v^k)(n_3) = n_2$  and  $\Gamma(v^k)(n_1) = n_4$ . Hence  $\lambda_1 = (e; n_1, n_4), \rho_1 = (e; n_3, n_2)$ . If  $w_2 = w$  then  $\Gamma(w)(n_1) = n_2$  and  $\Gamma(w)(n_3) = n_4$ . Since  $(n_1, n_2) \neq (n_3, n_4)$  and  $\Gamma(w) = (2, 3, \theta)(4, 1, \theta)$ ,  $n_1 = 4, n_2 = 1, n_3 = 2, n_4 = 3$  or  $n_1 = 2, n_2 = 3, n_3 = 4, n_4 = 1$ . Now as  $\Gamma(v^k)(n_3) = n_2$ ,  $\Gamma(v^k)(n_1) = n_4$  and  $\Gamma(v^k)(2) = \theta$  for  $k > 1$ ,  $v^k = v$  and thus  $Y = Y_n$ . So we may assume that there exists  $1 \leq l \leq n$  such that  $w_2 = v^l$ . Similarly we have  $\Gamma(v^l)(n_1) = n_2$  and  $\Gamma(v^l)(n_3) = n_4$ . It easily can be verified that  $n_4 = n_1 - k = n_3 - l, n_2 = n_1 - l = n_3 - k$ . Then  $k - l = l - k$  and thus  $k = l$ . Hence  $n_4 = n_2, n_1 = n_3$ , a contradiction.

Finally suppose that  $w_1 = w$ . As  $\Gamma(w_1)(n_3) = n_2$ ,  $\Gamma(w_1)(n_1) = n_4$ ,  $\Gamma(w) = (2, 3, \theta)(4, 1, \theta)$  and  $(n_1, n_2) \neq (n_3, n_4)$ ,  $\{x, y\} = \{(e; 4, 3), (e; 2, 1)\}$  and thus  $\rho_1, \lambda_1 \in \{(e; 2, 3), (e; 4, 1)\}$ . Now as  $\Gamma(\rho_1 w_2 \lambda_1)$  and  $\Gamma(\lambda_1 w_2 \rho_1)$  are non-zero, it follows that  $(2, 1, \theta) \subseteq \Gamma(w_2), (4, 3, \theta) \subseteq \Gamma(w_2)$ . Since  $\Gamma(v^k)(2) = \theta$  for  $k > 1$  and  $\Gamma(w)(2) = 3$ , one then obtains that  $w_2 = v$ . Therefore  $Y = Y_n$ .

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